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MIXED RESOLUTIONS AND SIMPLICIAL SECTIONS

BY

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ABSTRACT

We introduce the notions of mixed resolutions and simplicial sections, and prove a theorem relating them. This result is used (in another paper) to study deformation quantization in algebraic geometry.

0. Introduction

Let $\mathbb{K}$ be a field of characteristic 0. In this paper we present several technical results about the geometry of $\mathbb{K}$-schemes. These results were discovered in the course of work on deformation quantization in algebraic geometry, and they play a crucial role in [Ye3]. This role will be explained at the end of the introduction. The idea behind the constructions in this paper can be traced back to old work of Bott [Bo, HY].

Let $\pi: Z \to X$ be a morphism of $\mathbb{K}$-schemes, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an open covering of X. A **simplicial section** $\boldsymbol{\sigma}$ of π, based on the covering $\boldsymbol{U}$, consists of a family of morphisms $\sigma_{\boldsymbol{i}}: \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to Z$, where $\boldsymbol{i} = (i_0, \ldots, i_q)$ is a multi-index; $\boldsymbol{\Delta}^q_{\mathbb{K}}$ is the q-dimensional geometric simplex; and $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$. The morphisms $\sigma_{\boldsymbol{i}}$ are required to be compatible with π and to satisfy simplicial relations. See Definition 5.1 for details. An important example of a simplicial section is mentioned at the end of the introduction.

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Another notion we introduce is that of **mixed resolution**. Here we assume the $\mathbb{K}$-scheme X is smooth and separated, and each of the open sets $U_{(i)}$ in the covering $\boldsymbol{U}$ is affine. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we define its mixed resolution $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$. This is a complex of sheaves on X, concentrated in non-negative degrees. As the name suggests, this resolution mixes two distinct types of resolutions: a de Rham type resolution which is related to the sheaf $\mathcal{P}_X$ of principal parts of X and its Grothendieck connection, and a simplicial-Čech type resolution which is related to the covering $\boldsymbol{U}$. The precise definition is too complicated to state here— see Section 4.

Let $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$ denote the abelian category of bounded below complexes of quasi-coherent $\mathcal{O}_X$-modules. For any $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$ the mixed resolution $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is defined by totalizing the double complex $\bigoplus_{p,q} \operatorname{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p)$. The derived category of $\mathbb{K}$-modules is denoted by $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

Theorem 0.1: *Let X be a smooth separated $\mathbb{K}$-scheme, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an affine open covering of X.*

(1) *There is a functorial quasi-isomorphism*

$$\mathcal{M} \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$$

for $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$.

(2) *Given $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$, the canonical morphism*

$$\Gamma(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})) \to \mathrm{R}\Gamma(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}))$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ is an isomorphism.

(3) *The quasi-isomorphism in part (1) induces a functorial isomorphism $\Gamma(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})) \cong \mathrm{R}\Gamma(X, \mathcal{M})$ in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.*

This is repeated as Theorem 4.15. Note that part (3) is a formal consequence of parts (1) and (2).

A useful corollary of the theorem is the following (Corollary 4.16). Suppose $\mathcal{M}$ and $\mathcal{N}$ are two complexes in $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$, and ϕ: $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N})$ is a $\mathbb{K}$-linear quasi-isomorphism. Then

$$\Gamma(X, \phi) \colon \Gamma(X, \operatorname{Mix}_{\boldsymbol{U}} \mathcal{M})) \to \Gamma(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N}))$$

is a quasi-isomorphism.

Here is the connection between simplicial sections and mixed resolutions.

THEOREM 0.2: *Let X be a smooth separated $\mathbb{K}$-scheme, let $\pi\colon Z \to X$ be a morphism of schemes, and let $\boldsymbol{U}$ be an affine open covering of X. Suppose $\boldsymbol{\sigma}$ is a simplicial section of π based on $\boldsymbol{U}$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules, and let*

$$\phi\colon \prod_{i=1}^{r} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous $\mathcal{O}_Z$-multilinear sheaf morphism on Z. Then there is an induced $\mathbb{K}$-multilinear sheaf morphism

$$\boldsymbol{\sigma}^*(\phi)\colon \prod_{i=1}^{r} \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_i) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$$

on X.

In the theorem, the continuity and the complete pullback $\pi^{\widehat{*}}$ refer to the dir-inv structures on these sheaves, which are explained in Section 1. A more detailed statement is Theorem 5.2.

Let us explain, in vague terms, how Theorem 0.2, or rather Theorem 5.2, is used in the paper [Ye3]. Let X be a smooth separated n-dimensional $\mathbb{K}$-scheme. As we know from the work of Kontsevich [Ko], there are two important sheaves of DG Lie algebras on X, namely the sheaf $\mathcal{T}_{\mathrm{poly},X}$ of poly derivations, and the sheaf $\mathcal{D}_{\mathrm{poly},X}$ of poly differential operators. Suppose $\boldsymbol{U}$ is some affine open covering of X. The inclusions $\mathcal{T}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$ and $\mathcal{D}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ are then quasi-isomorphisms of sheaves of DG Lie algebras (cf. Theorem 0.1). The goal is to find an L_∞ quasi-isomorphism

$$\Psi\colon \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

between these sheaves of DG Lie algebras. Having such an L_∞ quasi-isomorphism pretty much implies the solution of the deformation quantization problem for X.

Let $\mathrm{Coor}\, X$ denote the coordinate bundle of X. This is an infinite dimensional bundle over X, endowed with an action of the group $\mathrm{GL}_n(\mathbb{K})$. Let $\mathrm{LCC}\, X$ be the quotient bundle $\mathrm{Coor}\, X/\mathrm{GL}_n(\mathbb{K})$. In [Ye4] we proved that if the covering $\boldsymbol{U}$ is fine enough (the condition is that each open set $U_{(i)}$ admits an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$), then the projection $\pi\colon \mathrm{LCC}\, X \to X$ admits a simplicial section $\boldsymbol{\sigma}$.

Now the universal deformation formula of Kontsevich [Ko] gives rise to a continuous L_∞ quasi-isomorphism

$$\mathcal{U}\colon \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$$

on LCC X. This means that there is a sequence of continuous $\mathcal{O}_{\mathrm{LCC}\,X}$-multilinear sheaf morphisms

$$\mathcal{U}_r\colon \prod^r \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}),$$

$r \geq 1$, satisfying very complicated identities. Using Theorem 5.2 we obtain a sequence of multilinear sheaf morphisms

$$\boldsymbol{\sigma}^*(\mathcal{U}_r)\colon \prod^r \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

on X. After twisting these morphisms suitably (this is needed due to the presence of the Grothendieck connection; cf. [Ye2]) we obtain the desired L_∞ quasi-isomorphism Ψ.

We believe that mixed resolutions, and the results of this paper, shall have additional applications in algebraic geometry (e.g. algebro-geometric versions of results on index theorems in differential geometry, cf. [NT]; or a proof of Kontsevich's famous yet unproved claim on Hochschild cohomology of a scheme [Ko, Claim 8.4]).

1. Review of Dir-Inv Modules

We begin the paper with a review of the concept of dir-inv structure, which was introduced in [Ye2]. A dir-inv structure is a generalization of an adic topology.

Let C be a commutative ring. We denote by $\mathsf{Mod}\,C$ the category of C-modules.

Definition 1.1:

(1) Let $M \in \mathsf{Mod}\,C$. An **inv module structure** on M is an inverse system $\{\mathrm{F}^i M\}_{i\in\mathbb{N}}$ of C-submodules of M. The pair $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ is called an **inv C-module**.

(2) Let $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i\in\mathbb{N}})$ be two inv C-modules. A function $\phi\colon M \to N$ (C-linear or not) is said to be **continuous** if for every $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\mathrm{F}^{i'} M) \subset \mathrm{F}^i N$.

(3) Define $\mathsf{Inv}\,\mathsf{Mod}\,C$ to be the category whose objects are the inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

There is a full and faithful embedding of categories $\mathsf{Mod}\,C \hookrightarrow \mathsf{Inv}\,\mathsf{Mod}\,C$, $M \mapsto (M, \{\ldots, 0, 0\})$.

Recall that a directed set is a partially ordered set J with the property that for any $j_1, j_2 \in J$ there exists $j_3 \in J$ such that $j_1, j_2 \leq j_3$.

Definition 1.2:

(1) Let $M \in \mathsf{Mod}\, C$. A **dir-inv module structure** on M is a direct system $\{\mathrm{F}_j M\}_{j \in J}$ of C-submodules of M, indexed by a nonempty directed set J, together with an inv module structure on each $\mathrm{F}_j M$, such that for every $j_1 \leq j_2$ the inclusion $\mathrm{F}_{j_1} M \hookrightarrow \mathrm{F}_{j_2} M$ is continuous. The pair $(M, \{\mathrm{F}_j M\}_{j \in J})$ is called a **dir-inv C-module**.

(2) Let $(M, \{\mathrm{F}_j M\})_{j \in J}$ and $(N, \{\mathrm{F}_k N\}_{k \in K})$ be two dir-inv C-modules. A function ϕ: $M \to N$ (C-linear or not) is said to be **continuous** if for every $j \in J$ there exists $k \in K$ such that $\phi(\mathrm{F}_j M) \subset \mathrm{F}_k N$, and ϕ: $\mathrm{F}_j M \to \mathrm{F}_k N$ is a continuous function between these two inv C-modules.

(3) Define $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$ to be the category whose objects are the dir-inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

An inv C-module M can be endowed with a dir-inv module structure $\{\mathrm{F}_j M\}_{j \in J}$, where $J := \{0\}$ and $\mathrm{F}_0 M := M$. Thus we get a full and faithful embedding $\mathsf{Inv}\,\mathsf{Mod}\, C \hookrightarrow \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$.

Inv modules and dir-inv modules come in a few "flavors": trivial, discrete and complete. A **discrete inv module** is one which is isomorphic, in $\mathsf{Inv}\,\mathsf{Mod}\, C$, to an object of $\mathsf{Mod}\, C$ (via the canonical embedding above). A **complete inv module** is an inv module $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ such that the canonical map $M \to \lim_{\leftarrow i} \mathrm{M}/F^i M$ is bijective. A **discrete** (resp., **complete**) **dir-inv module** is one which is isomorphic, in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$, to a dir-inv module $(M, \{\mathrm{F}_j M\}_{j \in J})$, where all the inv modules $\mathrm{F}_j M$ are discrete (resp., complete), and the canonical map $\lim_{j \to} \mathrm{F}_j M \to M$ in $\mathsf{Mod}\, C$ is bijective. A **trivial dir-inv module** is one which is isomorphic to an object of $\mathsf{Mod}\, C$. Discrete dir-inv modules are complete, but there are also other complete modules, as the next example shows.

Example 1.3: Assume C is noetherian and $\mathfrak{c}$-adically complete for some ideal $\mathfrak{c}$. Let M be a finitely generated C-module, and define $\mathrm{F}^i M := \mathfrak{c}^{i+1} M$. Then $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ is called the **$\mathfrak{c}$-adic inv structure**, and $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is a complete inv module. Next consider an arbitrary C-module M. We take $\{\mathrm{F}_j M\}_{j \in J}$ to be the collection of finitely generated C-submodules of M. This dir-inv module structure on M is called the **$\mathfrak{c}$-adic dir-inv structure**. Again $(M, \{\mathrm{F}_j M\}_{j \in J})$ is a complete dir-inv C-module. Note that a finitely generated C-module M is discrete as inv module if and only if $\mathfrak{c}^i M = 0$ for $i \gg 0$; and a C-module is discrete as dir-inv module if and only if it is a direct limit of discrete finitely generated modules.

The category $\mathsf{Dir\,Inv\,Mod}\, C$ is additive. Given a collection $\{M_k\}_{k \in K}$ of dir-inv modules, the direct sum $\bigoplus_{k \in K} M_k$ has a structure of dir-inv module, making it into the coproduct of $\{M_k\}_{k \in K}$ in the category $\mathsf{Dir\,Inv\,Mod}\, C$. Note that if the index set K is infinite and each M_k is a nonzero discrete inv module, then $\bigoplus_{k \in K} M_k$ is a discrete dir-inv module which is not trivial. The tensor product $M \otimes_C N$ of two dir-inv modules is again a dir-inv module. There is a completion functor $M \mapsto \widehat{M}$. (Warning: if M is complete then $\widehat{M} = M$, but it is not known if $\widehat{M}$ is complete for arbitrary M.) The completed tensor product is $M \widehat{\otimes}_C N := \widehat{M \otimes_C N}$. Completion commutes with direct sums: if $M \cong \bigoplus_{k \in K} M_k$ then $\widehat{M} \cong \bigoplus_{k \in K} \widehat{M}_k$. See [Ye2] for full details.

A **graded dir-inv module** (or graded object in $\mathsf{Dir\,Inv\,Mod}\, C$) is a direct sum $M = \bigoplus_{k \in \mathbb{Z}} M_k$, where each M_k is a dir-inv module. A **DG algebra** in $\mathsf{Dir\,Inv\,Mod}\, C$ is a graded dir-inv module $A = \bigoplus_{k \in \mathbb{Z}} A^k$, together with continuous C-(bi)linear functions μ: $A \times A \to A$ and d: $A \to A$, which make A into a DG C-algebra. If A is a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, and $\mathfrak{g}$ is a DG Lie Algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, then $A \widehat{\otimes}_C \mathfrak{g}$ is a DG Lie Algebra in $\mathsf{Dir\,Inv\,Mod}\, C$.

Let A be a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\, C$. A **DG A-module** in $\mathsf{Dir\,Inv\,Mod}\, C$ is a graded object M in $\mathsf{Dir\,Inv\,Mod}\, C$, together with continuous C-(bi)linear functions μ: $A \times M \to M$ and d: $M \to M$, which make M into a DG A-module in the usual sense. A **DG A-module Lie algebra** in $\mathsf{Dir\,Inv\,Mod}\, C$ is a DG Lie algebra $\mathfrak{g}$ in $\mathsf{Dir\,Inv\,Mod}\, C$, together with a continuous C-bilinear function μ: $A \times \mathfrak{g} \to \mathfrak{g}$, such that such that $\mathfrak{g}$ becomes a DG A-module, and

$$[a_1 \gamma_1, a_2 \gamma_2] = (-1)^{i_2 j_1} a_1 a_2 [\gamma_1, \gamma_2]$$

for all $a_k \in A^{i_k}$ and $\gamma_k \in \mathfrak{g}^{j_k}$.

All the constructions above can be geometrized. Let $(Y, \mathcal{O})$ be a commutative ringed space over $\mathbb{K}$, i.e., Y is a topological space and $\mathcal{O}$ is a sheaf of commutative $\mathbb{K}$-algebras on Y. We denote by $\mathsf{Mod}\, \mathcal{O}$ the category of $\mathcal{O}$-modules on Y.

Example 1.4: Geometrizing Example 1.3, let $\mathfrak{X}$ be a noetherian formal scheme, with defining ideal $\mathcal{I}$. Then any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is an inv $\mathcal{O}_{\mathfrak{X}}$-module, with system of submodules $\{\mathcal{I}^{i+1} \mathcal{M}\}_{i \in \mathbb{N}}$, and $\mathcal{M} \cong \widehat{\mathcal{M}}$; cf. [EGA-I]. We call an $\mathcal{O}_{\mathfrak{X}}$-module **dir-coherent** if it is the direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Any dir-coherent module is quasi-coherent, but it is not known if the converse is true. At any rate, a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is a dir-inv $\mathcal{O}_{\mathfrak{X}}$-module, where we take $\{\mathrm{F}_j \mathcal{M}\}_{j \in J}$ to be the collection of coherent submodules of $\mathcal{M}$.

Any dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is then a complete dir-inv module. This dir-inv module structure on $\mathcal{M}$ is called the **$\mathcal{I}$-adic dir-inv structure**. Note that a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is discrete as inv module if and only if $\mathcal{I}^i\mathcal{M} = 0$ for $i \gg 0$; and a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is discrete as dir-inv module if and only if it is a direct limit of discrete coherent modules.

If $f\colon (Y', \mathcal{O}') \to (Y, \mathcal{O})$ is a morphism of ringed spaces and $\mathcal{M} \in \mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then there is an obvious structure of dir-inv $\mathcal{O}'$ -module on $f^*\mathcal{M}$, and we define $f^{\widehat{*}}\mathcal{M} := \widehat{f^*\mathcal{M}}$. If $\mathcal{M}$ is a graded object in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then the inverse images $f^*\mathcal{M}$ and $f^{\widehat{*}}\mathcal{M}$ are graded objects in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. If $\mathcal{G}$ is an algebra (resp., a DG algebra) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then $f^*\mathcal{G}$ and $f^{\widehat{*}}\mathcal{G}$ are algebras (resp., DG algebras) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. Given $\mathcal{N} \in \mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$ there is an obvious dir-inv $\mathcal{O}$-module structure on $f_*\mathcal{N}$.

Example 1.5: Let $(Y, \mathcal{O})$ be a ringed space and $V \subset Y$ an open set. For a dir-inv $\mathcal{O}$-module $\mathcal{M}$ there is an obvious way to make $\Gamma(V, \mathcal{M})$ into a dir-inv $\Gamma(V, \mathcal{O})$-module. If $\mathcal{M}$ is a complete inv $\mathcal{O}$-module then $\Gamma(V, \mathcal{M})$ is a complete inv $\Gamma(V, \mathcal{O})$-module. If V is quasi-compact and $\mathcal{M}$ is a complete dir-inv $\mathcal{O}$-module, then $\Gamma(V, \mathcal{M})$ is a complete dir-inv $\Gamma(V, \mathcal{O})$-module.

2. Complete Thom-Sullivan Cochains

From here on $\mathbb{K}$ is a field of characteristic 0. Let us begin with some abstract notions about cosimplicial modules and their normalizations, following [HS] and [HY]. We use the notation $\mathsf{Mod}\,\mathbb{K}$ and $\mathsf{DGMod}\,\mathbb{K}$ for the categories of $\mathbb{K}$-modules and DG (differential graded) $\mathbb{K}$-modules respectively.

Let $\mathbf{\Delta}$ denote the category with objects the ordered sets $[q] := \{0, 1, \ldots, q\}$, $q \in \mathbb{N}$. The morphisms $[p] \to [q]$ are the order preserving functions, and we write $\mathbf{\Delta}_p^q := \mathrm{Hom}_{\mathbf{\Delta}}([p], [q])$. The i-th co-face map $\partial^i\colon [p] \to [p+1]$ is the injective function that does not take the value i; and the i-th co-degeneracy map $\mathrm{s}^i\colon [p] \to [p-1]$ is the surjective function that takes the value i twice. All morphisms in $\mathbf{\Delta}$ are compositions of various ∂^i and s^i.

An element of $\mathbf{\Delta}_p^q$ may be thought of as a sequence $\boldsymbol{i} = (i_0, \ldots, i_p)$ of integers with $0 \le i_0 \le \cdots \le i_p \le q$. Given $\boldsymbol{i} \in \mathbf{\Delta}_q^m$, $\boldsymbol{j} \in \mathbf{\Delta}_m^p$ and $\alpha \in \mathbf{\Delta}_p^q$, we sometimes write $\alpha_*(\boldsymbol{i}) := \boldsymbol{i} \circ \alpha \in \mathbf{\Delta}_p^m$ and $\alpha^*(\boldsymbol{j}) := \alpha \circ \boldsymbol{j} \in \mathbf{\Delta}_m^q$.

Let C be some category. A **cosimplicial object** in C is a functor $C\colon \mathbf{\Delta} \to \mathsf{C}$. We shall usually refer to the cosimplicial object as $C = \{C^p\}_{p\in\mathbb{N}}$, and for any $\alpha \in \mathbf{\Delta}_p^q$ the corresponding morphism in C will be denoted by $\alpha^*\colon C^p \to C^q$. A **simplicial object** in C is a functor $C\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$. The notation for a simplicial object will be $C = \{C_p\}_{p\in\mathbb{N}}$ and $\alpha_*\colon C_q \to C_p$.

Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial $\mathbb{K}$-module. The **standard normalization** of M is the DG module $\mathrm{N}M$ defined as follows:

$$\mathrm{N}^q M := \bigcap_{i=0}^{q-1} \mathrm{Ker}(\mathrm{s}^i \colon M^q \to M^{q-1}).$$

The differential is $\partial := \sum_{i=0}^{q+1} (-1)^i \partial^i \colon \mathrm{N}^q M \to \mathrm{N}^{q+1} M$. We get a functor $\mathrm{N} \colon \mathbf{\Delta}\,\mathsf{Mod}\,\mathbb{K} \to \mathsf{DGMod}\,\mathbb{K}$.

For any q let $\mathbf{\Delta}^q_{\mathbb{K}}$ be the **geometric q-dimensional simplex**

$$\mathbf{\Delta}^q_{\mathbb{K}} := \mathrm{Spec}\ \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1).$$

The i-th vertex of $\mathbf{\Delta}^q_{\mathbb{K}}$ is the $\mathbb{K}$-rational point x such that $t_i(x) = 1$ and $t_j(x) = 0$ for all $j \neq i$. We identify the vertices of $\mathbf{\Delta}^q_{\mathbb{K}}$ with the ordered set $[q] = \{0, 1, \ldots, q\}$. For any $\alpha \colon [p] \to [q]$ in $\mathbf{\Delta}$ there is a unique linear morphism $\alpha \colon \mathbf{\Delta}^p_{\mathbb{K}} \to \mathbf{\Delta}^q_{\mathbb{K}}$ extending it, and in this way $\{\mathbf{\Delta}^q_{\mathbb{K}}\}_{q \in \mathbb{N}}$ is a cosimplicial scheme.

For a $\mathbb{K}$-scheme X we write $\Omega^p(X) := \Gamma(X, \Omega^p_{X/\mathbb{K}})$. Taking $X := \mathbf{\Delta}^q_{\mathbb{K}}$ we have a super-commutative associative unital DG $\mathbb{K}$-algebra $\Omega(\mathbf{\Delta}^q_{\mathbb{K}}) = \bigoplus_{p \in \mathbb{N}} \Omega^p(\mathbf{\Delta}^q_{\mathbb{K}})$, that is generated as $\mathbb{K}$-algebra by the elements $t_0, \ldots, t_q, \mathrm{d}t_0, \ldots, \mathrm{d}t_q$. The collection $\{\Omega(\mathbf{\Delta}^q_{\mathbb{K}})\}_{q \in \mathbb{N}}$ is a simplicial DG algebra, namely a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of DG $\mathbb{K}$-algebras.

In [HY], we made use of the Thom-Sullivan normalization $\tilde{\mathrm{N}}M$ of a cosimplicial $\mathbb{K}$-module M. For some applications (specifically, [Ye3]) a complete version of this construction is needed. Recall that for $M, N \in \mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ we can define the complete tensor product $N \widehat{\otimes} M$. The $\mathbb{K}$-modules $\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}})$ are always considered as discrete inv modules, so $\Omega(\mathbf{\Delta}^l_{\mathbb{K}})$ is a discrete dir-inv DG $\mathbb{K}$-algebra.

Definition 2.1: Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial dir-inv $\mathbb{K}$-module, namely each $M^q \in \mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, and the morphisms $\alpha^* \colon M^p \to M^q$, for $\alpha \in \mathbf{\Delta}^q_p$, are continuous $\mathbb{K}$-linear homomorphisms. Let

$$\widehat{\tilde{\mathrm{N}}}^q M \subset \prod_{l=0}^{\infty} (\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} M^l) \tag{2.2}$$

be the submodule consisting of all sequences

$$(u_0, u_1, \ldots), \quad \text{with } u_l \in \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} M^l,$$

such that

$$(\mathbf{1} \otimes \alpha^*)(u_k) = (\alpha_* \otimes \mathbf{1})(u_l) \in \Omega^q(\mathbf{\Delta}^k_{\mathbb{K}}) \widehat{\otimes} M^l, \tag{2.3}$$

for all $k, l \in \mathbb{N}$ and all $\alpha \in \mathbf{\Delta}^l_k$. Define a coboundary operator $\partial\colon \widehat{\tilde{\mathrm{N}}}^q M \to \widehat{\tilde{\mathrm{N}}}^{q+1} M$ using the exterior derivative $\mathrm{d}\colon \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \to \Omega^{q+1}(\mathbf{\Delta}^l_{\mathbb{K}})$. The resulting DG $\mathbb{K}$-module $(\widehat{\tilde{\mathrm{N}}} M, \partial)$ is called the **complete Thom-Sullivan normalization of** M.

The $\mathbb{K}$-module $\widehat{\tilde{\mathrm{N}}} M = \bigoplus_{q \in \mathbb{N}} \widehat{\tilde{\mathrm{N}}}^q M$ is viewed as an abstract module. We obtain a functor

$$\widehat{\tilde{\mathrm{N}}}\colon \mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K} \to \mathsf{DGMod}\,\mathbb{K}.$$

Remark 2.4: In case each M^l is a discrete dir-inv module one has

$$\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} M^l = \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \otimes M^l,$$

and therefore $\widehat{\tilde{\mathrm{N}}} M = \tilde{\mathrm{N}} M$.

The standard normalization $\mathrm{N} M$ also makes sense here, via the forgetful functor $\mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K} \to \mathbf{\Delta}\,\mathsf{Mod}\,\mathbb{K}$. The two normalizations $\widehat{\tilde{\mathrm{N}}}$ and N are related as follows. Let $\int_{\mathbf{\Delta}^l}\colon \Omega(\mathbf{\Delta}^l_{\mathbb{K}}) \to \mathbb{K}$ be the $\mathbb{K}$-linear map of degree $-l$ defined by integration on the compact real l-dimensional simplex, namely $\int_{\mathbf{\Delta}^l} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_l = \frac{1}{l!}$ etc. Suppose each dir-inv module M^l is complete, so that using [Ye2, Proposition 1.5] we get a functorial $\mathbb{K}$-linear homomorphism

$$\int_{\mathbf{\Delta}^l} : \Omega(\mathbf{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} M^l \to \mathbb{K} \widehat{\otimes} M^l \cong M^l.$$

Proposition 2.5: *Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial dir-inv $\mathbb{K}$-module, with all dir-inv modules M^q complete. Then the homomorphisms $\int_{\mathbf{\Delta}^l}$ induce a quasi-isomorphism*

$$\int_{\mathbf{\Delta}} : \widehat{\tilde{\mathrm{N}}} M \to \mathrm{N} M$$

in $\mathsf{DGMod}\,\mathbb{K}$.

Proof: This is a complete version of [HY, Theorem 1.12]. Let $\mathbf{\Delta}^l$ be the simplicial set $\mathbf{\Delta}^l := \mathrm{Hom}_{\mathbf{\Delta}}(-, [l])$; so its set of p-simplices is $\mathbf{\Delta}^l_p$. Define C_l to be the algebra of normalized cochains on $\mathbf{\Delta}^l$, namely

$$C_l := \mathrm{N}\,\mathrm{Hom}_{\mathsf{Sets}}(\mathbf{\Delta}^l, \mathbb{K}) \cong \mathrm{Hom}_{\mathsf{Sets}}(\mathbf{\Delta}^{l,\mathrm{nd}}, \mathbb{K}).$$

Here $\mathbf{\Delta}^{l,\mathrm{nd}}$ is the (finite) set of nondegenerate simplices, i.e., those sequences $\boldsymbol{i} = (i_0, \ldots, i_p)$ satisfying $0 \leq i_0 < \cdots < i_p \leq l$. As explained in [HY, Appendix A], we have simplicial DG algebras $C = \{C_l\}_{l \in \mathbb{N}}$ and $\Omega(\mathbf{\Delta}_{\mathbb{K}}) = \{\Omega(\mathbf{\Delta}^l_{\mathbb{K}})\}_{l \in \mathbb{N}}$, and a homomorphism of simplicial DG modules $\rho\colon \Omega(\mathbf{\Delta}_{\mathbb{K}}) \to C$.

It turns out (due to Bousfield-Gugenheim) that ρ is a homotopy equivalence in $\mathbf{\Delta}^{\mathrm{op}}\,\mathsf{DGMod}\,\mathbb{K}$, i.e., there are simplicial homomorphisms ϕ: $C \to \Omega(\mathbf{\Delta}_{\mathbb{K}})$, h: $C \to C$ and h': $\Omega(\mathbf{\Delta}_{\mathbb{K}}) \to \Omega(\mathbf{\Delta}_{\mathbb{K}})$ such that $\mathbf{1} - \rho \circ \phi = h \circ \mathrm{d} + \mathrm{d} \circ h$ and $\mathbf{1} - \phi \circ \rho = h' \circ \mathrm{d} + \mathrm{d} \circ h'$.

Now, for $M = \{M^q\} \in \mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ and $N = \{N_q\} \in \mathbf{\Delta}^{\mathrm{op}}\,\mathsf{Mod}\,\mathbb{K}$, let $N \widehat{\otimes}_{\leftarrow} M$ be the complete version of [HY, formula (A.1)], so that, in particular, $\Omega(\mathbf{\Delta}_{\mathbb{K}}) \widehat{\otimes}_{\leftarrow} M \cong \widehat{\tilde{\mathrm{N}}} M$ and $C \widehat{\otimes}_{\leftarrow} M \cong \mathrm{N} M$. Moreover,

$$\rho \widehat{\otimes}_{\leftarrow} \mathbf{1}_M = \int_{\mathbf{\Delta}} : \widehat{\tilde{\mathrm{N}}} M \to \mathrm{N} M.$$

It follows that $\int_{\mathbf{\Delta}}$ is a homotopy equivalence in $\mathsf{DGMod}\,\mathbb{K}$. ∎

Suppose $A = \{A^q\}_{q \in \mathbb{N}}$ is a cosimplicial DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ (not necessarily associative or commutative). This is a pretty complicated object: for every q we have a DG algebra $A^q = \bigoplus_{i \in \mathbb{Z}} A^{q,i}$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. For every $\alpha \in \mathbf{\Delta}_p^q$ there is a continuous DG algebra homomorphism α^*: $A^p \to A^q$, and the α^* have to satisfy the simplicial relations.

Both $\widehat{\tilde{\mathrm{N}}} A$ and $\mathrm{N} A$ are DG algebras. For $\widehat{\tilde{\mathrm{N}}} A$, the DG algebra structure comes from that of the DG algebras $\Omega(\mathbf{\Delta}_{\mathbb{K}}^l) \widehat{\otimes} A^l$, via the embeddings (2.2). In case each A^l is an associative super-commutative unital DG $\mathbb{K}$-algebra, then so is $\widehat{\tilde{\mathrm{N}}} A$. Likewise for DG Lie algebras. (The algebra $\mathrm{N} A$, with its Alexander–Whitney product, is very noncommutative.)

Assume that each $A^{q,i}$ is complete, so that the integral $\int_{\mathbf{\Delta}} : \widehat{\tilde{\mathrm{N}}} A \to \mathrm{N} A$ is defined. This is not a DG algebra homomorphism. However:

Proposition 2.6: *Suppose $A = \{A^q\}_{q \in \mathbb{N}}$ is a cosimplicial DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, with all A^q complete. Then the homomorphisms $\int_{\mathbf{\Delta}^l}$ induce an isomorphism of graded algebras*

$$\mathrm{H}\Big(\int_{\mathbf{\Delta}}\Big) : \mathrm{H} \widehat{\tilde{\mathrm{N}}} A \xrightarrow{\simeq} \mathrm{H} \mathrm{N} A.$$

Proof: This is a complete variant of [HY, Theorem 1.13]. The proof is identical, after replacing "$\otimes$" with "$\widehat{\otimes}$" where needed; cf., proof of the previous proposition. ∎

Remark 2.7: If A is associative then presumably $\int_{\mathbf{\Delta}}$ extends to an A_∞ quasi-isomorphism $\widehat{\tilde{\mathrm{N}}} A \to \mathrm{N} A$.

3. Commutative Čech Resolutions

In this section $\mathbb{K}$ is a field of characteristic 0 and X is a noetherian topological space. We denote by $\mathbb{K}_X$ the constant sheaf $\mathbb{K}$ on X. We will be interested in the category $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, whose objects are sheaves of $\mathbb{K}$-modules on X with dir-inv structures. Note that any open set $V \subset X$ is quasi-compact.

Let $X = \bigcup_{i=0}^{m} U_{(i)}$ be an open covering, which we denote by $\boldsymbol{U}$. For any $\boldsymbol{i} = (i_0, \ldots, i_q) \in \boldsymbol{\Delta}_q^m$ define $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$, and let $g_{\boldsymbol{i}}\colon U_{\boldsymbol{i}} \to X$ be the inclusion. Given a dir-inv $\mathbb{K}_X$-module $\mathcal{M}$ and natural number q we define a sheaf

$$\mathrm{C}^q(\boldsymbol{U}, \mathcal{M}) := \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_q^m} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} \mathcal{M}.$$

This is a finite product. For an open set $V \subset X$ we then have

$$\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M})) = \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_q^m} \Gamma(V \cap U_{\boldsymbol{i}}, \mathcal{M}).$$

For any $\boldsymbol{i}$ the $\mathbb{K}$-module $\Gamma(V \cap U_{\boldsymbol{i}}, \mathcal{M})$ has a dir-inv structure. Hence, $\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))$ is a dir-inv $\mathbb{K}$-module. If $\mathcal{M}$ happens to be a complete dir-inv $\mathbb{K}_X$-module then $\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))$ is a complete dir-inv $\mathbb{K}$-module, since each $V \cap U_{\boldsymbol{i}}$ is quasi-compact.

Keeping V fixed we get a cosimplicial dir-inv $\mathbb{K}$-module $\{\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))\}_{q \in \mathbb{N}}$. Applying the functors N^q and $\widehat{\tilde{\mathrm{N}}}{}^q$ we obtain $\mathbb{K}$-modules $\mathrm{N}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$ and $\widehat{\tilde{\mathrm{N}}}{}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$. As V varies these become presheaves of $\mathbb{K}$-modules, and are denoted by $\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ and $\widehat{\tilde{\mathrm{N}}}{}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$.

Recall that a simplex $\boldsymbol{i} = (i_0, \ldots, i_q)$ is nondegenerate if $i_0 < \cdots < i_q$. Let $\boldsymbol{\Delta}_q^{m,\mathrm{nd}}$ be the set of non-degenerate simplices inside $\boldsymbol{\Delta}_q^m$.

Lemma 3.1: *For every q the presheaves*

$$\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})\colon V \mapsto \mathrm{N}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$$

and

$$\widehat{\tilde{\mathrm{N}}}{}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})\colon V \mapsto \widehat{\tilde{\mathrm{N}}}{}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$$

are sheaves. There is a functorial isomorphism of sheaves

$$\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \cong \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_q^{m,\mathrm{nd}}} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} \mathcal{M}, \tag{3.2}$$

and functorial embeddings of sheaves

$$\widehat{\tilde{\mathrm{N}}}{}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \hookrightarrow \prod_{l \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_l^m} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} (\Omega^q(\boldsymbol{\Delta}_{\mathbb{K}}^l) \widehat{\otimes} \mathcal{M}) \tag{3.3}$$

and

$$\mathcal{M} \hookrightarrow \widehat{\tilde{\mathrm{N}}}^0\mathrm{C}(\boldsymbol{U}, \mathcal{M}). \tag{3.4}$$

Proof: Since $\{\mathrm{C}^q(\boldsymbol{U}, \mathcal{M})\}_{q\in\mathbb{N}}$ is a cosimplicial sheaf we get the isomorphism (3.2).

As for $\widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$, consider the sheaf $\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\mathcal{M}$ on X. Take any open set $V \subset X$ and $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$. Since $V \cap U_{\boldsymbol{i}}$ is quasi-compact we have

$$\begin{aligned}\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\Gamma(V \cap U_{\boldsymbol{i}}, \mathcal{M}) &\cong \Gamma(V \cap U_{\boldsymbol{i}}, \Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\mathcal{M}) \\ &= \Gamma(V, g_{\boldsymbol{i}*}g_{\boldsymbol{i}}^{-1}(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\mathcal{M})).\end{aligned}$$

By Definition 2.1 there is an exact sequence of presheaves on X:

$$\begin{aligned}0 \to \widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \to &\prod_{l\in\mathbb{N}}\prod_{\boldsymbol{i}\in\boldsymbol{\Delta}^m_l} g_{\boldsymbol{i}*}g_{\boldsymbol{i}}^{-1}(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\mathcal{M}) \\ &\xrightarrow{\mathbf{1}\otimes\alpha^* - \alpha_*\otimes\mathbf{1}} \prod_{k,l\in\mathbb{N}}\prod_{\alpha\in\boldsymbol{\Delta}^l_k}\prod_{\boldsymbol{i}\in\boldsymbol{\Delta}^m_l} g_{\boldsymbol{i}*}g_{\boldsymbol{i}}^{-1}(\Omega^q(\boldsymbol{\Delta}^k_{\mathbb{K}})\widehat{\otimes}\mathcal{M}).\end{aligned}$$

Since the presheaves in the middle and on the right are actually sheaves, it follows that $\widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is also a sheaf.

Finally the embedding (3.4) comes from the embeddings $\mathcal{M} \hookrightarrow \Omega^0(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}\mathcal{M}$, $w \mapsto 1 \otimes w$. ∎

Thus we have complexes of sheaves $\mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ and $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. There are functorial homomorphisms $\mathcal{M} \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ and $\mathcal{M} \to \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. Note that the complex $\Gamma(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M}))$ is nothing but the usual global Čech complex of $\mathcal{M}$, for the covering $\boldsymbol{U}$.

Definition 3.5: The complex $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is called the **commutative Čech resolution** of $\mathcal{M}$.

The reason for the name is that $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$ is a sheaf of super-commutative DG algebras, as can be seen from the next lemma.

Lemma 3.6: *Suppose $\mathcal{M}_1, \ldots, \mathcal{M}_r$ and $\mathcal{N}$ are dir-inv $\mathbb{K}_X$-modules, and $q_1, \ldots, q_r \in \mathbb{N}$. Let $q := q_1 + \cdots + q_r$. Suppose that for every $l \in \mathbb{N}$ and $\boldsymbol{i} \in \boldsymbol{\Delta}^m_l$ we are given $\mathbb{K}$-multilinear sheaf maps*

$$\begin{aligned}\phi_{q_1,\ldots,q_r,\boldsymbol{i}}\colon (\Omega^{q_1}(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}(\mathcal{M}_1|_{U_{\boldsymbol{i}}})) \times \cdots \times (\Omega^{q_r}(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}(\mathcal{M}_r|_{U_{\boldsymbol{i}}})) \\ \longrightarrow \Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}})\widehat{\otimes}(\mathcal{N}|_{U_{\boldsymbol{i}}})\end{aligned}$$

that are continuous (for the dir-inv module structures), and are compatible with the simplicial structure as in Definition 2.1. Then there are unique $\mathbb{K}$*-multilinear sheaf maps*

$$\phi_{q_1,\ldots,q_r}\colon \hat{\tilde{\mathrm{N}}}^{q_1}\mathrm{C}(\boldsymbol{U},\mathcal{M}_1)\times\cdots\times\hat{\tilde{\mathrm{N}}}^{q_r}\mathrm{C}(\boldsymbol{U},\mathcal{M}_r)\to\hat{\tilde{\mathrm{N}}}^{q}\mathrm{C}(\boldsymbol{U},\mathcal{N}),$$

that commute with the embeddings(3.3)*.*

Proof: Direct verification. ∎

Lemma 3.7: *Let* $\mathcal{M}_1,\ldots,\mathcal{M}_r,\mathcal{N}$ *be dir-inv* $\mathbb{K}_X$*-modules, and* $\phi\colon \prod\mathcal{M}_i\to\mathcal{N}$ *a continuous* $\mathbb{K}$*-multilinear sheaf homomorphism. Then there is an induced homomorphism of complexes of sheaves*

$$\phi\colon \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M}_1)\otimes\cdots\otimes\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M}_r)\to\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{N}).$$

Proof: Use Lemma 3.6. ∎

In particular, if $\mathcal{M}$ is a dir-inv $\mathcal{O}_X$-module, then $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})$ is a DG $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{O}_X)$-module.

If $\mathcal{M}=\bigoplus_p\mathcal{M}^p$ is a graded dir-inv $\mathbb{K}_X$-module, we define

$$\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})^i:=\bigoplus\nolimits_{p+q=i}\hat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U},\mathcal{M}^p)$$

and

$$\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M}):=\bigoplus\nolimits_i\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})^i.$$

Due to Lemma 3.7, if $\mathcal{M}$ is a complex in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, then $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})$ is also a complex (in $\mathsf{Mod}\,\mathbb{K}_X$), and there is a functorial homomorphism of complexes $\mathcal{M}\to\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})$.

Theorem 3.8: *Let* X *be a noetherian topological space, with open covering* $\boldsymbol{U}=\{U_{(i)}\}_{i=0}^m$*. Let* $\mathcal{M}$ *be a bounded below complex in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$*, and assume each* $\mathcal{M}^p$ *is a complete dir-inv* $\mathbb{K}_X$*-module. Then:*

(1) *For any open set* $V\subset X$ *the homomorphism*

$$\Gamma\Big(V,\int_{\boldsymbol{\Delta}}\Big)\colon \Gamma(V,\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M}))\to\Gamma(V,\mathrm{NC}(\boldsymbol{U},\mathcal{M})),$$

is a quasi-isomorphism of complexes of $\mathbb{K}$*-modules.*

(2) *There are functorial quasi-isomorphism of complexes of* $\mathbb{K}_X$*-modules*

$$\mathcal{M}\to\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U},\mathcal{M})\xrightarrow{\int_{\boldsymbol{\Delta}}}\mathrm{NC}(\boldsymbol{U},\mathcal{M}).$$

Proof: (1) Lemma 3.1 and Proposition 2.5 imply that for any p the homomorphism of complexes

$$\Gamma\Big(V, \int_{\boldsymbol{\Delta}}\Big) \colon \Gamma(V, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}^p)) \to \Gamma(V, \mathrm{NC}(\boldsymbol{U}, \mathcal{M}^p)),$$

is a quasi-isomorphism. Now use the standard filtration argument (the complexes in question are all bounded below).
(2) From (1) we deduce that

$$\Gamma\Big(V, \int_{\boldsymbol{\Delta}}\Big) \colon \Gamma(V, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})) \to \Gamma(V, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})) \tag{3.9}$$

is a quasi-isomorphism. Hence,

$$\int_{\boldsymbol{\Delta}} \colon \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$$

is a quasi-isomorphism of complexes of sheaves.

It is a known fact that $\mathcal{M}^p \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M}^p)$ is a quasi-isomorphism of sheaves (see, [Ha] Lemma 4.2). Again, this implies that $\mathcal{M} \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ is a quasi-isomorphism. And, therefore, the homomorphism $\mathcal{M} \to \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ coming from (3.4) is also a quasi-isomorphism. ∎

Now, let us look at a separated noetherian formal scheme $\mathfrak{X}$. Let $\mathcal{I}$ be some defining ideal of $\mathfrak{X}$, and let X be the scheme with structure sheaf $\mathcal{O}_X := \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$. So $\mathfrak{X}$ and X have the same underlying topological space. Recall that a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module which is the union of its coherent submodules.

Corollary 3.10: *Let $\mathfrak{X}$ be a noetherian separated formal scheme over $\mathbb{K}$, with defining ideal $\mathcal{I}$ and underlying topological space X. Let $\boldsymbol{U} = \{U_{(i)}\}_{i=0}^m$ be an affine open covering of X. Let $\mathcal{M}$ be a bounded below complex of sheaves of $\mathbb{K}$-modules on X. Assume each $\mathcal{M}^p$ is a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module, and the coboundary operators $\mathcal{M}^p \to \mathcal{M}^{p+1}$ are continuous for the $\mathcal{I}$-adic dir-inv structures (but not necessarily $\mathcal{O}_{\mathfrak{X}}$-linear). Then:*

(1) *The canonical morphism*

$$\Gamma(X, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})) \to \mathrm{R}\Gamma(X, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}))$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ *is an isomorphism.*

(2) *There is a functorial isomorphism*

$$\Gamma(X, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})) \cong \mathrm{R}\Gamma(X, \mathcal{M})$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

Proof: (1) Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \widehat{\widetilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})) & \xrightarrow{\Gamma(X, \int_{\boldsymbol{\Delta}})} & \Gamma(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma(X, \widehat{\widetilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})) & \xrightarrow{\mathrm{R}\Gamma(X, \int_{\boldsymbol{\Delta}})} & \mathrm{R}\Gamma(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$, in which the vertical arrows are the canonical morphisms. By part (1) of Theorem 3.8 (with $V = X$) the top arrow is a quasi-isomorphism. And by part (2) the bottom arrow is an isomorphism. Hence it is enough to prove that the right vertical arrow is an isomorphism.

Using a filtration argument we may assume that $\mathcal{M}$ is a single dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module. Now $\Gamma(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M}))$ is the usual Čech resolution of the sheaf $\mathcal{M}$ with respect to the covering $\boldsymbol{U}$ (cf., (3.2)). So it suffices to prove that for all q and $\boldsymbol{i} \in \boldsymbol{\Delta}_q^{m,\mathrm{nd}}$ the sheaves $g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} \mathcal{M}$ are $\Gamma(X, -)$-acyclic.

First, let us assume $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Let $\mathfrak{U}_{\boldsymbol{i}}$ be the open formal subscheme of $\mathfrak{X}$ supported on $U_{\boldsymbol{i}}$. Then $g_{\boldsymbol{i}}^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{U}_{\boldsymbol{i}}}$-module, and both $g_{\boldsymbol{i}} \colon \mathfrak{U}_{\boldsymbol{i}} \to \mathfrak{X}$ and $\mathfrak{U}_{\boldsymbol{i}} \to \operatorname{Spec} \mathbb{K}$ are affine morphisms. By [EGA-I, Theorem 10.10.2] it follows that $g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} \mathcal{M} = \mathrm{R} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} \mathcal{M}$, and also

$$\Gamma(U_{\boldsymbol{i}}, g_{\boldsymbol{i}}^{-1}\mathcal{M}) = \mathrm{R}\Gamma(U_{\boldsymbol{i}}, g_{\boldsymbol{i}}^{-1}\mathcal{M}) \cong \mathrm{R}\Gamma(X, \mathrm{R}g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1}\mathcal{M}) \cong \mathrm{R}\Gamma(X, g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1}\mathcal{M}).$$

We conclude that $\mathrm{H}^j(X, g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1}\mathcal{M}) = 0$ for all $j > 0$.

In the general case when $\mathcal{M}$ is a direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules we still get $\mathrm{H}^j(X, g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1}\mathcal{M}) = 0$ for all $j > 0$.

(2) By part (2) of Theorem 3.8 we get a functorial isomorphism $\mathrm{R}\Gamma(X, \mathcal{M}) \cong \mathrm{R}\Gamma(X, \widehat{\widetilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}))$. Now use part (1) above. ∎

4. Mixed Resolutions

In this section $\mathbb{K}$ is s field of characteristic 0 and X is a finite type $\mathbb{K}$-scheme.

Let us begin be recalling the definition of the sheaf of principal parts $\mathcal{P}_X$ from [EGA IV]. Let $\Delta \colon X \to X^2 = X \times_{\mathbb{K}} X$ be the diagonal embedding. By completing X^2 along $\Delta(X)$ we obtain a noetherian formal scheme $\mathfrak{X}$, and $\mathcal{P}_X := \mathcal{O}_{\mathfrak{X}}$. The two projections $\mathrm{p}_i \colon X^2 \to X$ give rise to two ring homomorphisms $\mathrm{p}_i^* \colon \mathcal{O}_X \to \mathcal{P}_X$. We view $\mathcal{P}_X$ as a left (resp., right) $\mathcal{O}_X$-module via p_1^* (resp., p_2^*).

Recall that a connection ∇ on an $\mathcal{O}_X$-module $\mathcal{M}$ is a $\mathbb{K}$-linear sheaf homomorphism $\nabla\colon \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ satisfying the Leibniz rule $\nabla(fm) = \mathrm{d}(f) \otimes m + f\nabla(m)$ for local sections $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$.

Definition 4.1: Consider the de Rham differential $\mathrm{d}_{X^2/X}\colon \mathcal{O}_{X^2} \to \Omega^1_{X^2/X}$ relative to the morphism $\mathrm{p}_2\colon X^2 \to X$. Since $\Omega^1_{X^2/X} \cong \mathrm{p}_1^*\Omega^1_X = \mathrm{p}_1^{-1}\Omega^1_X \otimes_{\mathrm{p}_1^{-1}\mathcal{O}_X} \mathcal{O}_{X^2}$, we obtain a $\mathbb{K}$-linear homomorphism $\mathrm{d}_{X^2/X}\colon \mathcal{O}_{X^2} \to \mathrm{p}_1^*\Omega^1_X$. Passing to the completion along the diagonal $\Delta(X)$ we get a connection of $\mathcal{O}_X$-modules

$$\nabla_{\mathcal{P}}\colon \mathcal{P}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X, \tag{4.2}$$

called the **Grothendieck connection**.

Note that the connection $\nabla_{\mathcal{P}}$ is $\mathrm{p}_2^{-1}\mathcal{O}_X$-linear. It will be useful to describe $\nabla_{\mathcal{P}}$ on the level of rings. Let $U = \operatorname{Spec} C \subset X$ be an affine open set. Then

$$\Gamma(U, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega^1_C \otimes_C (\widehat{C \otimes C}) \cong \widehat{\Omega^1_C \otimes C},$$

is the I-adic completion, where $I := \operatorname{Ker}(C \otimes C \to C)$. And

$$\nabla_{\mathcal{P}}\colon \widehat{C \otimes C} \to \widehat{\Omega^1_C \otimes C}$$

is the completion of $\mathrm{d} \otimes \mathbf{1}\colon C \otimes C \to \Omega^1_C \otimes C$.

The connection $\nabla_{\mathcal{P}}$ of (4.2) induces differential operators of left $\mathcal{O}_X$-modules

$$\nabla_{\mathcal{P}}\colon \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \to \Omega^{i+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

for all $i \geq 0$, by the rule

$$\nabla_{\mathcal{P}}(\alpha \otimes b) = \mathrm{d}(\alpha) \otimes b + (-1)^i \alpha \wedge \nabla_{\mathcal{P}}(b). \tag{4.3}$$

Theorem 4.4: *Assume X is a smooth n-dimensional $\mathbb{K}$-scheme. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then the sequence of sheaves on X,*

$$\begin{aligned} 0 \to \mathcal{M} \xrightarrow{m \mapsto 1 \otimes m} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \cdots \xrightarrow{\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}} \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to 0, \end{aligned} \tag{4.5}$$

is exact.

Proof: The proof is similar to that of [Ye1, Theorem 4.5]. We may restrict to an affine open set $U = \operatorname{Spec} B \subset X$ that admits an étale coordinate system $\boldsymbol{s} = (s_1, \ldots, s_n)$, i.e., $\mathbb{K}[\boldsymbol{s}] \to B$ is an étale ring homomorphism. It will be convenient to have another copy of B, which we call C; so that $\Gamma(U, \mathcal{P}_X) = \widehat{B \otimes C}$, the

I-adic completion, where $I := \operatorname{Ker}(B \otimes C \to B)$. We shall identify B and C with their images inside $B \otimes C$, and denote the copy of the element s_i in C by r_i. Letting $t_i := r_i - s_i \in B \otimes C$ we then have $t_i = \tilde{s}_i = 1 \otimes s_i - s_i \otimes 1$ in our earlier notation. Note that $\Omega_{\mathbb{K}[\boldsymbol{s}]} \subset \Omega_B$ is a sub DG algebra, and $B \otimes_{\mathbb{K}[\boldsymbol{s}]} \Omega_{\mathbb{K}[\boldsymbol{s}]} \to \Omega_B$ is a bijection.

By definition

$$\Gamma(U, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega_B \otimes_B (\widehat{B \otimes C}) \cong \widehat{\Omega_B \otimes C}. \tag{4.6}$$

The differential $\nabla_{\mathcal{P}}$ on the left goes to the differential $\mathrm{d}_B \otimes \mathbf{1}_C$ on the right. Consider the sub DG algebra $\Omega_{\mathbb{K}[\boldsymbol{s}]} \otimes C \subset \Omega_B \otimes C$. We know that $\mathbb{K} \to \Omega_{\mathbb{K}[\boldsymbol{s}]}$ is a quasi-isomorphism; therefore, so is $C \to \Omega_{\mathbb{K}[\boldsymbol{s}]} \otimes C$.

Since $t_i + s_i = r_i \in C$, we see that $C[\boldsymbol{s}] = C[\boldsymbol{t}] \subset B \otimes C$. Therefore, we obtain C-linear isomorphisms

$$\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes C \cong \Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{s}] = \Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}].$$

So there is a commutative diagram

(4.7)

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & C & \longrightarrow & C[\boldsymbol{t}] & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}] & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots & \Omega^n_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & B \otimes C & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_B \otimes C & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots & \Omega^n_B \otimes C & \longrightarrow & 0
\end{array}$$

of C-modules. The top row is exact, and the vertical arrow are inclusions. Let us introduce a new grading on $\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}]$ as follows: $\deg(s_i) := 1$, $\deg(t_i) := 1$, $\deg(\mathrm{d}(s_i)) := 1$ and $\deg(c) := 0$ for every nonzero $c \in C$. Since $\nabla_{\mathcal{P}}(t_i) = -\mathrm{d}(s_i)$, we see that $\nabla_{\mathcal{P}}$ is homogeneous of degree 0, thus the top row in (4.7) is an exact sequence in the category $\mathsf{GrMod}\, C$ of graded C-modules. Now each term in this sequence is a free graded C-module, and, therefore, this sequence is split in $\mathsf{GrMod}\, C$.

The $\boldsymbol{t}$-adic inv structure on $C[\boldsymbol{t}]$ can be recovered from the grading, and this inv structure is the same as the I-adic inv structure on $B \otimes C$. Therefore, the completion is $\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] \cong \widehat{\Omega^p_B \otimes C}$. Thus, the diagram (4.7) is transformed to the commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & C & \longrightarrow & C[[\boldsymbol{t}]] & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots & \Omega^n_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & \widehat{B \otimes C} & \xrightarrow{\nabla_{\mathcal{P}}} & \widehat{\Omega^1_B \otimes C} & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots & \widehat{\Omega^n_B \otimes C} & \longrightarrow & 0
\end{array}$$

in which the top row is continuously C-linearly split and the vertical arrows are bijections. Hence, the bottom row is split exact. Comparing this to (4.6) we conclude that the sequence of right $\mathcal{O}_U$-modules

$$0 \to \mathcal{O}_U \xrightarrow{\mathrm{p}_2^*} \mathcal{P}_X|_U \xrightarrow{\nabla_{\mathcal{P}}} (\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X)|_U \xrightarrow{\nabla_{\mathcal{P}}} \cdots (\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X)|_U \to 0$$

is split exact.

It follows that for any $\mathcal{O}_X$-module $\mathcal{M}$ the sequence (4.5), when restricted to U, is split exact. ∎

Let us now fix an affine open covering $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ of X.

Let $\mathcal{I}_X = \operatorname{Ker}(\mathcal{P}_X \to \mathcal{O}_X)$. This is a defining ideal of the noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) := (X, \mathcal{P}_X)$. So $\mathcal{P}_X$ is an inv module over itself with the $\mathcal{I}_X$-adic inv structure. Given quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, the tensor product $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a dir-coherent $\mathcal{P}_X$-module, and so it has the $\mathcal{I}_X$-adic dir-inv structure. See Example 1.4. In particular,

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus\nolimits_{p \geq 0} \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

becomes a dir-inv $\mathbb{K}_X$-module.

Lemma 4.8: *$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a DG Ω_X-module in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, *with differential $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$.*

Proof: Since $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$ is a differential operator of $\mathcal{P}_X$-modules, it is continuous for the $\mathcal{I}_X$-adic dir-inv structure. See [Ye2, Proposition 2.3]. ∎

Henceforth, we will write $\nabla_{\mathcal{P}}$ instead of $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$.

Definition 4.9: Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. For any $p, q \in \mathbb{N}$ define

$$\operatorname{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) := \widehat{\tilde{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

The Grothendieck connection

$$\nabla_{\mathcal{P}} \colon \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

induces a homomorphism of sheaves

$$\nabla_{\mathcal{P}} \colon \operatorname{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \operatorname{Mix}^{p+1,q}_{\boldsymbol{U}}(\mathcal{M}).$$

We also have ∂: $\mathrm{Mix}_{\boldsymbol{U}}^{p,q}(\mathcal{M}) \to \mathrm{Mix}_{\boldsymbol{U}}^{p,q+1}(\mathcal{M})$. Define

$$\mathrm{Mix}_{\boldsymbol{U}}^{i}(\mathcal{M}) := \bigoplus\nolimits_{p+q=i} \mathrm{Mix}_{\boldsymbol{U}}^{p,q}(\mathcal{M}),$$
$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus\nolimits_{i} \mathrm{Mix}_{\boldsymbol{U}}^{i}(\mathcal{M})$$

and

$$\mathrm{d}_{\mathrm{mix}} := \partial + (-1)^q \nabla_{\mathcal{P}}\colon \mathrm{Mix}_{\boldsymbol{U}}^{p,q}(\mathcal{M}) \to \mathrm{Mix}_{\boldsymbol{U}}^{p+1,q} \oplus \mathrm{Mix}_{\boldsymbol{U}}^{p,q+1}(\mathcal{M}). \tag{4.10}$$

The complex $(\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}), \mathrm{d}_{\mathrm{mix}})$ is called the **mixed resolution of $\mathcal{M}$**.

There are functorial embeddings of sheaves

$$\mathcal{M} \subset \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \subset \widehat{\tilde{\mathrm{N}}}{}^0 \mathrm{C}(\boldsymbol{U}, \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \mathrm{Mix}_{\boldsymbol{U}}^{0,0}(\mathcal{M}) \tag{4.11}$$

and

$$\mathrm{Mix}_{\boldsymbol{U}}^{p,q}(\mathcal{M}) \subset \prod_{l \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_l^m} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1} (\Omega^q(\boldsymbol{\Delta}_{\mathbb{K}}^l) \widehat{\otimes} (\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})); \tag{4.12}$$

see Lemma 3.1.

Proposition 4.13:

(1) $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$ *is a sheaf of super-commutative associative unital DG* $\mathbb{K}$-*algebras. There are two* $\mathbb{K}$-*algebra homomorphisms*

$$\mathrm{p}_1^*, \mathrm{p}_2^*\colon \mathcal{O}_X \to \mathrm{Mix}_{\boldsymbol{U}}^0(\mathcal{O}_X).$$

(2) *Let* $\mathcal{M}$ *be a quasi-coherent* $\mathcal{O}_X$-*module. Then* $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ *is a left DG* $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-*module.*

(3) *If* $\mathcal{M}$ *is a locally free* $\mathcal{O}_X$-*module of finite rank then the multiplication map*

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$$

is an isomorphism.

Proof: By Lemmas 3.1 and 3.7. ∎

Note that $\mathrm{d}_{\mathrm{mix}} \circ \mathrm{p}_2^*\colon \mathcal{O}_X \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$ is zero, but $\mathrm{d}_{\mathrm{mix}} \circ \mathrm{p}_1^* \neq 0$.

Proposition 4.14: *Let* $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ *be quasi-coherent* $\mathcal{O}_X$-*modules. Suppose*

$$\phi\colon \prod_{i=1}^{r} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

is a continuous Ω_X-multilinear sheaf morphism of degree d. Then there is a unique $\mathbb{K}$-multilinear sheaf morphism of degree d

$$\widehat{\widetilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \phi)\colon \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_1) \times \cdots \times \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_r) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N}),$$

which is compatible with ϕ via the embedding (4.12).

Proof: This is an immediate consequence of Lemma 3.7. ∎

Suppose we are given $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$. Define

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})^i := \bigoplus\nolimits_{p+q=i} \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p)$$

with differential

$$\mathrm{d}_{\mathrm{mix}} + (-1)^q \mathrm{d}_{\mathcal{M}}\colon \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p) \to \mathrm{Mix}^{q+1}_{\boldsymbol{U}}(\mathcal{M}^p) \oplus \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^{p+1}).$$

Theorem 4.15: *Let X be a smooth separated $\mathbb{K}$-scheme, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an affine open covering of X.*

1. *There is a functorial quasi-isomorphism $\mathcal{M} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ for $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$.*
2. *Given $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$, the canonical morphism*

$$\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})) \to \mathrm{R}\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}))$$

   in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ is an isomorphism.
3. *The quasi-isomorphism in part(1) induces a functorial isomorphism $\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})) \cong \mathrm{R}\Gamma(X, \mathcal{M})$ in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.*

Proof: (1) Write $\mathcal{N} := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$. A filtration argument and Theorem 4.4 show that the inclusion $\mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism. Next we view $\mathcal{N}$ as a bounded below complex in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K}_X$. By Theorem 3.8(2) we have a quasi-isomorphism $\mathcal{N} \to \widehat{\widetilde{\mathrm{N}}}{}^q\mathrm{C}(\boldsymbol{U}, \mathcal{N}) = \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$.
(2) This is due to Corollary 3.10(1), applied to the formal scheme $(X, \mathcal{P}_X)$ and the complex $\mathcal{N}$ of dir-coherent $\mathcal{P}_X$-modules defined above.
(3) The assertion is an immediate consequence of parts (1) and (2). ∎

Corollary 4.16: *In the situation of the theorem, suppose $\mathcal{M}$ and $\mathcal{N}$ are in $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$ and $\phi\colon \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$ is a $\mathbb{K}$-linear quasi-isomorphism. Then*

$$\Gamma(X, \phi)\colon \Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})) \to \Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N}))$$

is a quasi-isomorphism.

Proof: Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})) & \xrightarrow{\Gamma(X,\phi)} & \Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})) & \xrightarrow{\mathrm{R}\Gamma(X,\phi)} & \mathrm{R}\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})), \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$. By part (2) of Theorem 4.15 the vertical arrows are isomorphisms. Since ϕ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K}_X)$, it follows that the bottom arrow is an isomorphism. ∎

Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and an integer i define

$$\mathrm{G}^i \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus\nolimits_{q \geq i} \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}).$$

Then $\{\mathrm{G}^i \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\}_{i \in \mathbb{Z}}$ is a descending filtration of $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ by subcomplexes, satisfying $\mathrm{G}^i \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) = \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ for $i \ll 0$ and $\bigcap_i \mathrm{G}^i \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) = 0$. For any i define

$$\mathrm{gr}^i_{\mathrm{G}} \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \mathrm{G}^i \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) / \mathrm{G}^{i+1} \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}).$$

The functor

$$\mathrm{gr}^i_{\mathrm{G}} \,\mathrm{Mix}_{\boldsymbol{U}} : \mathsf{QCoh}\, \mathcal{O}_X \to \mathsf{Mod}\, \mathbb{K}_X$$

is additive, but we do not know whether it is exact. The next theorem asserts this in a very special case.

Consider the sheaves of DG Lie algebras $\mathcal{T}_{\mathrm{poly},X}$ and $\mathcal{D}_{\mathrm{poly},X}$ as complexes of quasi-coherent $\mathcal{O}_X$-modules (cf., [Ye3, Proposition 3.18]). According to [Ye1, Theorem 0.4] there is a quasi-isomorphism

$$\mathcal{U}_1 \colon \mathcal{T}_{\mathrm{poly},X} \to \mathcal{D}_{\mathrm{poly},X}.$$

Theorem 4.17: *For any i the homomorphism of complexes*

$$\mathrm{gr}^i_{\mathrm{G}} \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{U}_1) \colon \mathrm{gr}^i_{\mathrm{G}} \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{gr}^i_{\mathrm{G}} \,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

is a quasi-isomorphism.

Proof: Given a point $x \in X$ choose an affine open neighborhood V of x which admits anétale morphism $V \to \mathbf{A}^n_{\mathbb{K}}$. By [Ye2, Theorem 4.11], the map of complexes

$$\mathcal{U}_1|_V \colon \mathcal{T}_{\mathrm{poly},X}|_V \to \mathcal{D}_{\mathrm{poly},X}|_V$$

is a homotopy equivalence in $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_V)$. Since $\mathrm{gr}^i_{\mathrm{G}}\,\mathrm{Mix}_{\boldsymbol{U}}$ is an additive functor we see that $\mathrm{gr}^i_{\mathrm{G}}\,\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{U}_1)|_V$ is a quasi-isomorphism. ∎

Remark 4.18: We know very little about the structure of the sheaves, $\widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U},\mathcal{M})$, even when $\mathcal{M}=\mathcal{O}_X$. Cf. [HS].

5. Simplicial Sections

Let X be a $\mathbb{K}$-scheme, and let $X=\bigcup_{i=0}^m U_{(i)}$ be an open covering, with inclusions $g_{(i)}\colon U_{(i)}\to X$. We denote this covering by $\boldsymbol{U}$. For any multi-index $\boldsymbol{i}=(i_0,\ldots,i_q)\in\boldsymbol{\Delta}^m_q$ we write $U_{\boldsymbol{i}}:=\bigcap_{j=0}^q U_{(i_j)}$, and we define the scheme $U_q:=\coprod_{\boldsymbol{i}\in\boldsymbol{\Delta}^m_q}U_{\boldsymbol{i}}$. Given $\alpha\in\boldsymbol{\Delta}^q_p$ and $\boldsymbol{i}\in\boldsymbol{\Delta}^m_q$ there is an inclusion of open sets $\alpha_*\colon U_{\boldsymbol{i}}\to U_{\alpha_*(\boldsymbol{i})}$. These patch to a morphism of schemes $\alpha_*\colon U_q\to U_p$, making $\{U_q\}_{q\in\mathbb{N}}$ into a simplicial scheme. The inclusions $g_{(i)}\colon U_{(i)}\to X$ induce inclusions $g_{\boldsymbol{i}}\colon U_{\boldsymbol{i}}\to X$ and morphisms $g_q\colon U_q\to X$; and one has the relations $g_p\circ\alpha_*=g_q$ for any $\alpha\in\boldsymbol{\Delta}^q_p$.

Definition 5.1: Let $\pi\colon Z\to X$ be a morphism of $\mathbb{K}$-schemes. A **simplicial section of** π based on the covering $\boldsymbol{U}$ is a sequence of morphisms

$$\boldsymbol{\sigma}=\{\sigma_q\colon\boldsymbol{\Delta}^q_{\mathbb{K}}\times U_q\to Z\}_{q\in\mathbb{N}},$$

satisfying the following conditions.

(i) For any q the diagram

$$\begin{array}{ccc}\boldsymbol{\Delta}^q_{\mathbb{K}}\times U_q & \xrightarrow{\sigma_q} & Z\\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi}\\ U_q & \xrightarrow{g_q} & X\end{array}$$

is commutative.

(ii) For any $\alpha\in\boldsymbol{\Delta}^q_p$ the diagram

$$\begin{array}{ccccc} & & \boldsymbol{\Delta}^p_{\mathbb{K}}\times U_p & & \\ & \nearrow{\scriptstyle \mathbf{1}\times\alpha_*} & & \searrow{\scriptstyle \sigma_p} & \\ \boldsymbol{\Delta}^p_{\mathbb{K}}\times U_q & & & & Z\\ & \searrow{\scriptstyle \alpha^*\times\mathbf{1}} & & \nearrow{\scriptstyle \sigma_q} & \\ & & \boldsymbol{\Delta}^q_{\mathbb{K}}\times U_q & & \end{array}$$

is commutative.

Given a multi-index $\boldsymbol{i} \in \boldsymbol{\Delta}_q^m$ we denote by $\sigma_{\boldsymbol{i}}$ the restriction of σ_q to $\boldsymbol{\Delta}_{\mathbb{K}}^q \times U_{\boldsymbol{i}}$. See Figure 1 for an illustration.

As explained in the introduction, simplicial sections arise naturally in several contexts, including deformation quantization.

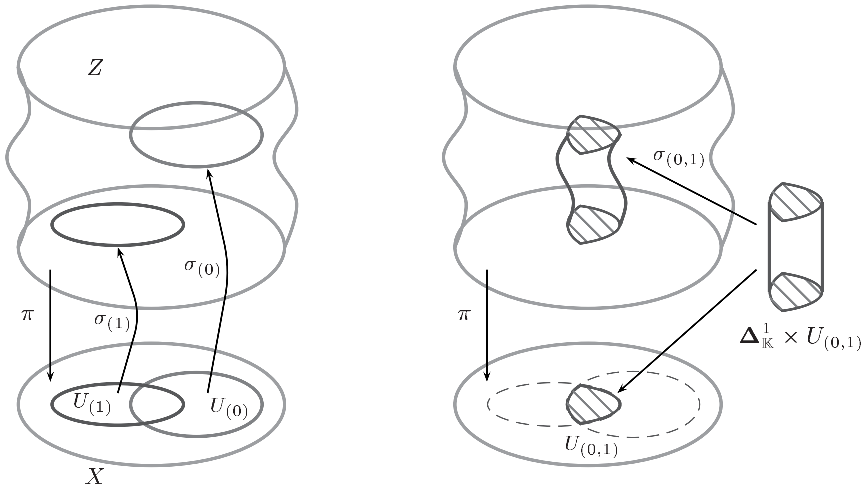

Figure 1. An illustration of a simplicial section $\boldsymbol{\sigma}$ based on an open covering $\boldsymbol{U} = \{U_{(i)}\}$. On the left we see two components of $\boldsymbol{\sigma}$ in dimension $q = 0$; and on the right we see one component in dimension $q = 1$.

Let A be an associative unital super-commutative DG $\mathbb{K}$-algebra. Consider homogeneous A-multilinear functions $\phi\colon M_1 \times \cdots \times M_r \to N$, where $M_1, \ldots, M_r$ and N are DG A-modules. There is an operation of composition for such functions: given functions $\psi_i\colon \prod_j L_{i,j} \to M_i$ the composition is

$$\phi \circ (\psi_1 \times \cdots \times \psi_r)\colon \prod_{i,j} L_{i,j} \to N.$$

There is also a summation operation: if $\phi_j\colon \prod_i M_i \to N$ are homogeneous of equal degree then so is their sum $\sum_j \phi_j$. Finally, let $\mathrm{d}\colon \prod_i M_i \to \prod_i M_i$ be the function

$$\mathrm{d}(m_1, \ldots, m_r) := \sum_{i=1}^{r} \pm(m_1, \ldots, \mathrm{d}(m_i), \ldots, m_r)$$

with Koszul signs. All the above can, of course, be sheafified, i.e., $\mathcal{A}$ is a sheaf of DG algebras on a scheme Z etc.

As before, let $\pi: Z \to X$ be a morphism if $\mathbb{K}$-schemes, and let $\boldsymbol{U} = \{U_{(i)}\}$ be an open covering of X. Suppose $\boldsymbol{\sigma}$ is a simplicial section of π based on $\boldsymbol{U}$. We consider Ω^p_X as a discrete inv $\mathbb{K}_X$-module, and $\Omega_X = \bigoplus_{p \geq 0} \Omega^p_X$ has the $\bigoplus$ dir-inv structure. Likewise for $\Omega_Z = \bigoplus_{p \geq 0} \Omega^p_Z$.

Suppose $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Then, as explained in Section 4, $\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a DG Ω_Z-module on Z, with the Grothendieck connection $\nabla_{\mathcal{P}}$. And $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is a DG $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-module on X, with differential $\mathrm{d}_{\mathrm{mix}}$.

Theorem 5.2: *Let $\pi: Z \to X$ be a morphism of schemes, and suppose $\boldsymbol{\sigma}$ is a simplicial section of π based on an open covering $\boldsymbol{U}$ of X. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules, and let*

$$\phi: \prod_{i=1}^{r} (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)) \to \Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous Ω_Z-multilinear sheaf morphism on Z of degree k. Then there is an induced $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-multilinear sheaf morphism of degree k

$$\boldsymbol{\sigma}^*(\phi): \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}_1) \times \cdots \times \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}_r) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N}),$$

on X with the following properties:

(i) *The assignment $\phi \mapsto \boldsymbol{\sigma}^*(\phi)$ respects the operations of composition and summation.*

(ii) *If $\phi = \pi^{\widehat{*}}(\phi_0)$ for some continuous Ω_X-multilinear morphism*

$$\phi_0: \prod_{i=1}^{r} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N},$$

then $\boldsymbol{\sigma}^*(\phi) = \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \phi_0)$.

(iii) *Assume that*

$$\nabla_{\mathcal{P}} \circ \phi - (-1)^k \phi \circ \nabla_{\mathcal{P}} = \psi$$

for some continuous Ω_Z-multilinear sheaf morphism

$$\psi: \prod_{i=1}^{r} (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)) \to \Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

of degree $k+1$. Then,

$$\mathrm{d}_{\mathrm{mix}} \circ \boldsymbol{\sigma}^*(\phi) - (-1)^k \boldsymbol{\sigma}^*(\phi) \circ \mathrm{d}_{\mathrm{mix}} = \boldsymbol{\sigma}^*(\psi).$$

Before the proof we need an auxiliary result.

Lemma 5.3: *Let A and B be complete DG algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, *and let $f^*\colon A \to B$ be a continuous DG algebra homomorphism. To any DG A-module M in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ *we assign the DG B-module $f^* M := B \widehat{\otimes}_A M$. Then to any continuous A-multilinear function $\phi\colon \prod_i M_i \to N$ we can assign a continuous B-multilinear function $f^*(\phi)\colon \prod_i f^*(M_i) \to f^*(N)$. This assignment is functorial in f^*, and respects the operations of composition and summation. If ϕ and ψ are such continuous A-multilinear functions, homogeneous of degrees k and $k+1$, respectively, and satisfying*

$$\mathrm{d} \circ \phi - (-1)^k \phi \circ \mathrm{d} = \psi,$$

then

$$\mathrm{d} \circ f^*(\phi) - (-1)^k f^*(\phi) \circ \mathrm{d} = f^*(\psi).$$

Proof: This is all straightforward, except perhaps the last assertion. For that, we make the calculations. By continuity and multilinearity it suffices to show that

$$(\mathrm{d} \circ f^*(\phi))(\beta) - (-1)^k (f^*(\phi) \circ \mathrm{d})(\beta) = f^*(\psi)(\beta),$$

for $\beta = (\beta_1, \ldots, \beta_r)$, with $\beta_i = b_i \otimes m_i$, $b_i \in B^{p_i}$ and $m_i \in M^{q_i}$. Then

$$\begin{aligned}(\mathrm{d} \circ f^*(\phi))(\beta) &= \mathrm{d}(\pm b_1 \ldots b_r \cdot \phi(m_1, \ldots, m_r)) \\ &= \pm \mathrm{d}(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot \mathrm{d}(\phi(m_1, \ldots, m_r))\end{aligned}$$

with Koszul signs. Since

$$\mathrm{d}(\beta_i) = \mathrm{d}(b_i) \otimes m_i \pm b_i \otimes \mathrm{d}(m_i),$$

we also have

$$\begin{aligned}(f^*(\phi) \circ \mathrm{d})(\beta) &= \sum_i \pm f^*(\phi)(\beta_1, \ldots, \mathrm{d}(\beta_i), \ldots, \beta_r) \\ &= \sum_i (\pm b_1 \cdots \mathrm{d}(b_i) \cdots b_r \cdot \phi(m_1, \ldots, m_r) \\ &\quad \pm b_1 \ldots b_r \cdot \phi(m_1, \ldots, \mathrm{d}(m_i) \cdots m_r)) \\ &= \pm\, \mathrm{d}(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot \phi(\mathrm{d}(m_1, \ldots, m_r)).\end{aligned}$$

Finally

$$f^*(\psi)(\beta) = \pm b_1 \cdots b_r \cdot \psi(m_1, \ldots, m_r),$$

and the signs all match up. ∎

Proof of the theorem: For a sequence of indices $\boldsymbol{i} = (i_0, \ldots, i_l) \in \boldsymbol{\Delta}^m_l$ let us introduce the abbreviation $Y_{\boldsymbol{i}} := \boldsymbol{\Delta}^l_{\mathbb{K}} \times U_{\boldsymbol{i}}$, and let $\mathrm{p}_2\colon Y_{\boldsymbol{i}} \to U_{\boldsymbol{i}}$ be the projection. The simplicial section $\boldsymbol{\sigma}$ restricts to a morphism $\sigma_{\boldsymbol{i}}\colon Y_{\boldsymbol{i}} \to Z$.

By Lemma 5.3, applied with respect to the DG algebra homomorphism $\sigma^*_{\boldsymbol{i}}\colon \sigma^{-1}_{\boldsymbol{i}}\Omega_Z \to \Omega_{Y_{\boldsymbol{i}}}$, there is an induced continuous $\Omega_{Y_{\boldsymbol{i}}}$-multilinear morphism

$$\begin{aligned}\sigma^*_{\boldsymbol{i}}(\phi)\colon \prod_{j=1}^{r} (\Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\sigma^{-1}_{\boldsymbol{i}}\Omega_Z} \sigma^{-1}_{\boldsymbol{i}} (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_j))) \\ \to \Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\sigma^{-1}_{\boldsymbol{i}}\Omega_Z} \sigma^{-1}_{\boldsymbol{i}} (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}))\end{aligned}$$

Now for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we have an isomorphism of dir-inv DG $\Omega_{Y_{\boldsymbol{i}}}$-modules

$$\Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\sigma^{-1}_{\boldsymbol{i}}\Omega_Z} \sigma^{-1}_{\boldsymbol{i}} (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})) \cong \Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\mathcal{O}_{Y_{\boldsymbol{i}}}} \mathrm{p}^*_2 (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

Under the DG algebra isomorphism $\mathrm{p}_{2*}\Omega_{Y_{\boldsymbol{i}}} \cong \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \otimes \Omega_{U_{\boldsymbol{i}}}$ there is a dir-inv DG module isomorphism

$$\mathrm{p}_{2*}(\Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\mathcal{O}_{Y_{\boldsymbol{i}}}} \mathrm{p}^*_2 (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})) \cong \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})|_{U_{\boldsymbol{i}}}.$$

Thus we obtain a family of morphisms

$$\begin{aligned}\sigma^*_{\boldsymbol{i}}(\phi)\colon \prod_{j=1}^{r} \Big(\Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_j)|_{U_{\boldsymbol{i}}}\Big) \\ \to \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \widehat{\otimes} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})|_{U_{\boldsymbol{i}}}),\end{aligned}$$

indexed by $\boldsymbol{i}$ and satisfying the simplicial relations. Now use Lemma 3.6 to obtain $\boldsymbol{\sigma}^*(\phi)$. Properties (i–iii) follow from Lemma 5.3. ∎

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